

Membrane paradigm and entropy of black holes in the Euclidean action approach

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The membrane paradigm approach to black holes fixes in the vicinity of the event horizon a fictitious surface, the stretched horizon, so that the spacetime outside remains unchanged and the spacetime inside is vacuum. Using this powerful method, several black hole properties have been found and settled, such as the horizon's viscosity, electrical conductivity, resistivity, as well as other properties. On the other hand the Euclidean action approach to black hole spacetimes has been very fruitful in understanding black hole entropy. Combining both the Euclidean action and membrane paradigm approaches a direct derivation of the black hole entropy is given. In the derivation it is considered that the only fields present are the gravitational and matter fields, with no electric field.

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I. INTRODUCTION

The viewpoint that the event horizon of a black hole acts for external observers as a membrane was initiated by Damour [1, 2] and continued by Thorne and collaborators [3–5]. This viewpoint together with its approach is called the membrane paradigm. The idea is to put in the vicinity of the event horizon a fictitious surface, the stretched horizon, so that the spacetime outside remains unchanged and the spacetime inside is vacuum nonsingular. The stretched horizon behaves as a membrane. To this membrane are attached proper boundary conditions so that all works as prescribed. This powerful approach enables one to deal with a timelike boundary (the stretched horizon) which is always more convenient in technical terms than the lightlike boundary of the event horizon. It was a formalism to help astrophysical calculations to be done in a more intuitive way [5]. After performing the calculations in the timelike membrane, which is located infinitesimally close to the true black hole horizon, one can always take the limit to a lightlike surface. The stretched horizon behaves as a membrane endowed with physical properties, and within the membrane approach several black hole properties have been found and settled, such as the horizon’s viscosity, electrical conductivity, resistivity, as well as other properties [1–5]. Also attempts to understand the Hawking bath and the Bekenstein-Hawking entropy has been dealt within this approach [6] (see also [5]). Extension of the approach to charged black holes has been done in [7, 8], and to black holes in $f(R)$ gravities in [9]. Properties of the stretched horizon as encoded in the quasinormal spectrum of black holes were explored in [10]. It has also connections to the fuzzball model [11]. Curiously, the membrane approach has found a great echo not in astrophysics and general relativity, but in string theory and related topics. Indeed, the approach has been useful in string theory and black holes [12, 13] and in the correspondence between fluids and gravity which was developed in the context of the duality between gauge theories and gravity [14].

Notwithstanding all these developments, the issue of the entropy of a black hole in the membrane paradigm approach has not been dealt with in the generality required for the importance of the subject. There are interesting discussions in [7, 8], where the relationship between the entropy and the horizon area is postulated, with an unknown coefficient of proportionality [7], and the specific Schwarzschild case in an asymptotically flat background is studied [8].

Now, several methods have been used to testify for the black hole entropy, from the original works [15, 16] to path integral methods [17–24] and other methods such as the quasiblack hole approach [25, 26]. In [20] it was first argued by York that to study thermodynamic black hole properties within a stable setting in the path integral approach one should immerse them in a thermal bath with a boundary, rather than in asymptotically flat spacetime as had been done in [17–19]. Of course the same remark applies to [8] and thus one should treat black hole entropy in the membrane paradigm in a full consistent manner.

The aim of the present paper is to give a simple and direct derivation of the Bekenstein-Hawking entropy, combining both the Euclidean action and membrane paradigm approaches. We do this for gravity coupled to matter alone, in the absence of an electric field. In Sec. II we write the general formulas for the metric, the temperature, the laws of thermodynamics using the path integral formalism, and the Euclidean action, and give the nomenclature used. In Sec. III we study the black hole entropy. We summarize the results found for the standard calculation and then we apply the formalism to the membrane paradigm approach. In Sec. IV we conclude.

II. GENERAL FORMULAS

A. Metric

Consider a static metric, not necessarily spherically symmetric. Assume that the metric is a solution of Einstein field equation, not coupled to any other long range field. Assume also that there is a compact body or a black hole. Then, at least in some vicinity of its boundary, or in the case of a black hole in some vicinity of the event horizon, the line element can be written as

$$ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j, \quad (1)$$

where N is the lapse function, t is the time coordinate, h_{ij} is the three-dimensional spatial metric and x^i, x^j represent the spatial coordinates. In Gaussian coordinates the three-dimensional line element $ds_3^2 = h_{ij} dx^i dx^j$ can be written as

$$ds_3^2 = dl^2 + \sigma_{ab} dx^a dx^b, \quad (2)$$

where l is the radial coordinate, x^a, x^b represent the angular coordinates (in the spherically-symmetric case) or their analogue for a more general metric, and σ_{ab} is a two-dimensional

metric. The whole metric in Gaussian coordinates is then

$$ds^2 = -N^2 dt^2 + dl^2 + \sigma_{ab} dx^a dx^b. \quad (3)$$

The metric functions N and σ_{ab} may have different forms for the inner and external parts. Suppose that there is a membrane somewhere. Then, the boundary marked by the membrane, the membrane boundary (mb), is assumed to be at $l = \text{const}$. For the membrane boundary, $dl = 0$, the line element can be written as

$$ds^2|_{\text{mb}} = \gamma_{mn} dx^m dx^n = -N_{\text{mb}}^2 dt^2 + \sigma_{ab} dx^a dx^b, \quad (4)$$

where γ_{mn} is the corresponding metric, and m, n represent time and angular coordinates (or their analogue).

B. Temperature

Assume also that the system is at a local Tolman temperature T given by

$$T = \frac{T_0}{N}, \quad (5)$$

where $T_0 = \text{constant}$. T_0 should be considered as the temperature at asymptotically flat infinity. It is useful to define the inverse temperature β as

$$\beta = \frac{1}{T}, \quad (6)$$

so that from Eq. (5) one has

$$\beta = N\beta_0. \quad (7)$$

C. Thermodynamics

There are many approaches to calculate the entropy and deal with the thermodynamics of a black hole. One can follow the original route where methods of field second quantization in a collapsing object are used to calculate the temperature T and then use the black hole laws to find the corresponding entropy [15, 16]. A more sophisticated approach is to use the Euclidean path integral approach to quantum gravity [17–19] and its developments [20–24] to obtain those thermodynamic properties. There are still other methods, see e.g., [25, 26].

The method we adopt here is the one that uses the path integral approach. The prescription implicit in this approach is that one can find the time evolution by calculating the amplitude to propagate a configuration between an initial and a final state. By Euclideanizing time and summing over a complete orthonormal basis of configurations the amplitude turns into the partition function Z , $Z = \sum \exp(-\beta E_n)$, of the field g at a temperature $T \equiv 1/\beta$, where E_n is the eigenenergy of the corresponding eigenstate. Implicit here is that one maintains the temperature fixed and so one uses the canonical ensemble. On the other hand, one can also represent the amplitude from one state to another using Feynman's prescription of a path integral over the action of the fields between the initial and final states. Since both prescriptions are equivalent, by Euclideanizing time one gets a representation for the partition functions in the path integral approach. Thus, in terms of the path integral formulation the partition function becomes $Z = \sum \exp(-\beta E_n) = \int D[g] \exp(-I)$, where I is the Euclidean action. The partition function for the field at temperature T is given by the path integral over the fields in a Euclidean spacetime. The first contributions to the Euclidean path integral are the most important. If the zeroth order contribution contributes is the most important, then $Z = \int D[g] \exp(-I) = \exp(-I)$, where I is now the zeroth order contribution.

The path-integral approach to the thermodynamics of black holes was originally developed by Hawking and collaborators, see e.g., [17–19]. In this approach the thermodynamical partition function is computed from the path integral in the saddle-point approximation. It was found that the Euclidean Schwarzschild black hole space is periodic in the direction of the imaginary time, with period β , and thus has temperature $T = 1/\beta$. Using the partition function and its relation to the thermodynamical potentials the thermodynamical laws as well as the entropy of black holes are obtained. It assumes that the partition function contains the zeroth order classical Euclidean Einstein action of a black hole as its leading term [19]. York extended the formalism for cavities of finite size [20] (see also [21–24]). In York's formalism the black hole is enclosed in a cavity with a finite radius. The boundary conditions are defined according to the thermodynamical ensemble under study. Given the boundary conditions and imposing the appropriate constraints, one can compute a reduced action suitable for doing black hole thermodynamics.

Now, in zeroth order one has $Z = \exp(-I)$. On the other hand one knows that the relation between the Helmholtz free energy F and the partition function Z is $\ln Z = -\beta F$.

So,

$$-I = \ln Z = -\beta F. \quad (8)$$

But the thermodynamic relation between the Helmholtz free energy F , the energy E , the temperature $T = 1/\beta$, and the entropy S is $F = E - TS$. Thus the Euclidean action I relates to E , β , and S as,

$$I = \beta E - S. \quad (9)$$

The energy is then given by

$$E = \frac{dI}{d\beta}. \quad (10)$$

So the formalism hinges on calculating the Euclidean action for the system in question. This is what we provide in the following.

D. Action

The total Euclidean action I represents the sum of the gravitational action I_g and the matter action I_{matter} ,

$$I = I_g + I_{\text{matter}}. \quad (11)$$

The gravitational action is

$$I_g = I_R + I_b, \quad (12)$$

where I_R is the bulk action and I_b is the boundary term action. The bulk action is given by

$$I_R = -\frac{1}{16\pi} \int d^4x \sqrt{-g} R, \quad (13)$$

with R being the Ricci scalar. Note that one can write

$$R = R_3 - \frac{2}{N} \Delta_3 N, \quad (14)$$

where R_3 is the spatial Ricci tensor in three dimensions, and Δ_3 is the Laplacian operator in three dimensions for the metric h_{ij} . The tt component of the Einstein equations, which can be viewed as a Hamiltonian constraint, gives us

$$R_3 = 16\pi\rho \quad (15)$$

where ρ is the energy density of the matter. The boundary term I_b is introduced for self-consistency of the variational procedure [18],

$$I_b = \frac{1}{8\pi} \int d^3x \sqrt{\gamma} (K - K_0). \quad (16)$$

Here, K is the extrinsic curvature of the three-dimensional boundary embedded in the four-dimensional spacetime and γ is the determinant of the γ_{mn} metric, see Eq. (4). The constant K_0 is usually chosen to make the action zero for the flat case. If one writes $K = -g^{ij}n_{i;j}$ in terms of the covariant derivatives of an outward normal vector n_i and takes into account that $\sqrt{-g} = N\sqrt{\sigma}$ where $\sigma = \det \sigma_{ab}$, one can obtain the known formula

$$K = k - \frac{N_{,i}n^i}{N}. \quad (17)$$

Here, k is the extrinsic curvature for the two-dimensional boundary surface embedded into the three-dimensional space.

The matter action is generically given by

$$I_{\text{matter}} = \beta_0 \int d^3x \sqrt{-g} \rho - S_{\text{matter}} \quad (18)$$

where $\beta_0 \equiv T_0^{-1}$, and S_{matter} is the matter entropy.

III. BLACK HOLE ENTROPY

A. Black hole entropy in the usual path-integral approach

1. Preliminaries

The calculation of the black hole entropy using the usual path-integral approach is by now standard. We refer to [17–24]. Below we mention some of the important results.

2. No black hole case

Let us suppose to begin with that there is no black hole. We assume that our system is situated inside some external boundary (eb). Then, taking into account equations (3)-(18) and using the Gauss theorem, we obtain

$$I_{\text{withoutbh}} = \int_{\text{eb}} d\sigma \beta \varepsilon - S_{\text{matter}} \quad (19)$$

where the integral is performed at an external boundary (eb), and ε is the spacetime energy density [23],

$$\varepsilon = \frac{k - K_0}{8\pi}. \quad (20)$$

Here $d\sigma$ is an element of area of the boundary and S_{matter} is the entropy of the matter. The formulas simplify slightly, if the boundary is an equipotential surface, i.e., T obeys $T = \text{constant}$ on it. As well $\beta = \text{constant}$ on the boundary. Then, (19) turns to $I = \beta E - S_{\text{matter}}$, where $E = \int d\sigma \varepsilon$ is the quasilocal energy.

3. Black hole case

Let us suppose now that there is a true black hole. In the Euclideanized manifold it is called a bolt. Then, taking into account equations (3)-(18) and using the Gauss theorem, we obtain (see [20, 21] for the spherically symmetric case and [22] for the general static one)

$$I_{\text{withbh}} = \int_{\text{eb}} d\sigma \beta \varepsilon - S_{\text{tot}}, \quad (21)$$

where,

$$S_{\text{tot}} = S_{\text{matter}} + \frac{A}{4}, \quad (22)$$

with A being the horizon area, $\frac{A}{4}$ is the Bekenstein-Hawking entropy, and S_{matter} is the matter entropy outside the horizon.

B. Black hole entropy in the membrane approach

1. Preliminaries

Now we want to show that the Bekenstein-Hawking entropy is reproduced within the membrane paradigm. Within the membrane paradigm one has a whole boundary b that consists of two pieces, the external boundary (eb), and the internal boundary or membrane boundary (mb). (a) The external boundary (eb): Sometimes, this boundary is chosen at infinity [18] (see also [8]). However, then one is faced with instabilities of the corresponding solutions [20]. Therefore, we do not impose such a requirement and consider an external boundary with a finite radius. (b) The membrane boundary (mb): This is the internal boundary, slightly above the horizon, such that the proper length l obeys $l \rightarrow 0$ when the boundary approaches the horizon.

2. No black hole case

One should be very careful in the choice of the boundary. If, say, the system is spherically symmetric, with a radius for the external boundary an external radius r_{eb} and a membrane radius r_{mb} , the physical results depend crucially on whether we impose boundary conditions on (1) $r_{\text{mb}} + \delta$ or (2) $r_{\text{mb}} - \delta$, with δ infinitesimal. Correspondingly, one should add the boundary term given in Eq. (16) in which the extrinsic curvature is calculated with respect either to (1) the geometry with $r > r_{\text{mb}}$, i.e., between r_{mb} and r_{eb} , or (2) the geometry outside the system, from $r < r_{\text{mb}}$.

In the first case, i.e., imposing boundary conditions on $r_{\text{mb}} + \delta$, repeating all calculations described above we obtain (19) where, however, the boundary term consists of two parts corresponding to the external and inner boundaries: The action is then,

$$I_{\text{withoutbh1}} = \left(\int_{\text{eb}} d\sigma \beta \varepsilon - \int_{\text{mb}} d\sigma \beta \varepsilon \right) - S_{\text{matter}} \quad (23)$$

where we took into account that on the inner boundary the outward normal is pointed in the opposite direction. Here, there is no term associated with the horizon at all. This is physically natural since we discarded from the very beginning the part of manifold that could contain a horizon.

In the second case, i.e., imposing boundary conditions on $r_{\text{mb}} - \delta$, the physical boundary is on the inner side, so there is no other boundary between the membrane surface and a horizon. The Euclidean action differs from (23) due to two other boundary terms, I_{-}^{surf} and I_{+}^{surf} , giving, $I_{\text{withoutbh2}} = I_{\text{withoutbh1}} + I_{-}^{\text{surf}} - I_{+}^{\text{surf}}$. More explicitly,

$$I_{\text{withoutbh2}} = \left(\int_{\text{eb}} d\sigma \beta \varepsilon - \int_{\text{mb}} d\sigma \beta \varepsilon \right) - S_{\text{matter}} + I_{-}^{\text{surf}} - I_{+}^{\text{surf}} \quad (24)$$

Here, the terms I_{-}^{surf} and I_{+}^{surf} have the form (16), where the subscript $+$ means that the corresponding quantity is calculated on the $+$ side of the membrane and the subscript $-$ means that the corresponding quantity is calculated on the $-$ side of the membrane, and where the integration is taken on the corresponding side of the surface. By construction, we assume that inside the membrane there is flat spacetime (vacuum). Therefore, in (16) $K = K_0$ and $I_{-}^{\text{surf}} = 0$. In I_{+}^{surf} , we take into account the formula (17). Then, the first term in (17) compensates the second term in the right hand side of (24) and we have

$$I_{\text{withoutbh2}} = \int_{\text{eb}} d\sigma \beta \varepsilon - S_{\text{matter}} - S_{\text{mb}}, \quad (25)$$

with

$$S_{\text{mb}} = \frac{\beta_0}{8\pi} \int_{\text{mb}} d\sigma \left(\frac{\partial N}{\partial n} \right)_+ , \quad (26)$$

where the subscript $+$ means that the corresponding quantity is calculated on the $+$ side of the membrane. The quantity β_0 is the inverse temperature at the membrane. The set given by Eqs. (25)-(26) is valid for any position of the membrane that separates the flat spacetime inside and the original geometry between the membrane and the external boundary.

3. Black hole case

To study the black hole case we should resort to the second case, i.e., imposing boundary conditions on $r_{\text{mb}} - \delta$, with the physical boundary being on the inner side and thus use Eqs. (25)-(26). When the position of the membrane moves towards the horizon, $N_{\text{mb}} \rightarrow 0$ where N_{mb} is the lapse function on the membrane, so $\beta \rightarrow 0$ as well due to (7). Apart from this, $\left(\frac{\partial N}{\partial n} \right)_+ \rightarrow \kappa$ where κ is the surface gravity. As it is constant on the horizon, it can be taken outside the integrand, with the result that $S = \frac{A}{4} \frac{T_{\text{H}}}{T_0}$ where $T_{\text{H}} = \frac{\kappa}{2\pi}$ is the Hawking temperature. In the state of thermal equilibrium we must have $T_0 = T_{\text{H}}$, so we have the final result

$$\lim_{l \rightarrow 0} S_{\text{mb}} = \frac{A}{4} . \quad (27)$$

Thus Eqs. (25)-(26) give

$$I_{\text{withbh}} = \int_{\text{eb}} d\sigma \beta \varepsilon - S_{\text{tot}} , \quad (28)$$

with

$$S_{\text{tot}} = S_{\text{matter}} + \frac{A}{4} , \quad (29)$$

where S_{matter} is the entropy outside. This is precisely the same formula as Eq. (21).

IV. CONCLUSION

We have shown that if one replaces a black hole horizon by a material membrane situated slightly above the horizon and consider the state of thermal equilibrium in the limit when this membrane approaches the horizon the Bekenstein-Hawking value is correctly reproduced. The essential ingredient is the posing of the correct boundary conditions on the membrane itself. Indeed, the boundary term in the action, necessary for a self-consistent variational

procedure, is precisely the term responsible for this entropy. Thus, both the standard Euclidean action for a true black hole (with no boundary term in the action on the horizon since the horizon is not a material surface) and the membrane paradigm give exactly the same result. It would be important to generalize further the present results to the cases with an electric charge and to the rotating case.

Another interesting task is the derivation of the black hole entropy from the quasiblack hole approach using the Euclidian action approach and comparison of it with what is done here within the membrane paradigm. The black hole entropy from the quasiblack hole approach using the first general law was obtained in [25, 26].

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- [1] T. Damour, “Black-hole eddy currents”, *Phys. Rev. D* **18**, 3598 (1978).
 - [2] T. Damour, “Surface effects in black hole physics”, in *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, ed. R. Ruffini, p. 587 (North-Holland, Amsterdam 1982).
 - [3] D. A. Macdonald and K. S. Thorne, “Black-hole electrodynamics: an absolute-space/universal-time formulation”, *Mon. Not. Roy. Astr. Soc.* **198**, 345 (1982).
 - [4] R. H. Price and K. S. Thorne, “Membrane viewpoint on black holes: Properties and evolution of the stretched horizon”, *Phys. Rev. D* **33**, 915 (1986).
 - [5] K. S. Thorne, R. H. Price, and D. A. Macdonald (eds.), *Black holes: The membrane paradigm*, (Yale University Press, London 1986).
 - [6] W. H. Zurek and K. S. Thorne, “Statistical mechanical origin of the entropy of a rotating, charged black hole”, *Phys. Rev. Lett.* **54**, 2171 (1985).

- [7] M. K. Parikh and F. Wilczek, “An action for black hole membranes”, Phys. Rev. D **58**, 064011 (1998); arXiv:hep-th/9907002.
- [8] M. K. Parikh, “Membrane horizons: The black hole’s new clothes”, hep-th/9907002 (1999).
- [9] S. Chatterjee, M. K. Parikh, S. Sarkar, “The black hole membrane paradigm in $f(R)$ gravity”; arXiv:1012.6040 [hep-th].
- [10] A. O. Starinets, “Quasinormal spectrum and the black hole membrane paradigm”, Phys. Lett. **B 670**, 442 (2009); arXiv:0806.3797 [hep-th].
- [11] S. D. Mathur, “Membrane paradigm realized?”, Gen. Rel. Grav. **42**, 2331 (2010); arXiv:1005.3555 [hep-th].
- [12] L. Susskind, L. Thorlacius and J. Uglum, “The stretched horizon and black hole complementarity”, Phys. Rev. D **48**, 3743 (1993); arXiv:hep-th/9306069.
- [13] L. Susskind and J. Lindesay, *An introduction to black holes, information and the string theory revolution: The holographic universe*, (World Scientific, Singapore 2003);
- [14] V. E. Hubeny, “The fluid/gravity correspondence: a new perspective on the membrane paradigm”; arXiv:1011.4948 [hep-th].
- [15] J. D. Bekenstein, “Black holes and entropy”, Phys. Rev. D **7**, 2333 (1973).
- [16] S. W. Hawking, “Particle creation by black holes”, Commun. Math. Phys. **43**, 199 (1975).
- [17] J. B. Hartle and S. W. Hawking “Path-integral derivation of black-hole radiance”, Phys. Rev. D **13**, 2188 (1976).
- [18] G. W. Gibbons and S. W. Hawking, “Action integrals and partition functions in quantum gravity”, Phys. Rev. D **15**, 2752 (1977).
- [19] S. W. Hawking, “The path-integral approach to quantum gravity”, in *General Relativity: an Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel [Paperback] (Cambridge University Press, Cambridge 1979), p. 746.
- [20] J. W. York, “Black-hole thermodynamics and the Euclidean Einstein action”, Phys. Rev. D **33**, 2092 (1986).
- [21] E. A. Martinez and J. W. York, “Additivity of the entropies of black holes and matter in equilibrium”, Phys. Rev. D **40**, 2124 (1989).
- [22] O. B. Zaslavskii, “Canonical ensemble for arbitrary configurations of self-gravitating systems”, Phys. Lett. A **152**, 463 (1991).
- [23] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the

- gravitational action”, *Phys. Rev. D* **47**, 1407 (1993).
- [24] C. Peça and J. P. S. Lemos, “Thermodynamics of Reissner-Nordstrom-anti-de Sitter black holes in the grand canonical ensemble”, *Phys. Rev. D* **59**, 124007 (1999); arXiv:gr-qc/9805004.
- [25] J. P. S. Lemos and O. B. Zaslavskii, “Entropy of quasiblack holes”, *Phys. Rev. D* **82**, 024029 (2010); arXiv:0904.1741 [gr-qc].
- [26] J. P. S. Lemos and O. B. Zaslavskii, “Entropy of extremal black holes from entropy of quasiblack holes”, *Phys. Lett. B* **695** 37 (2011); arXiv:1011.2768 [gr-qc].